Find $\lim_{x \to -\infty} \tanh x$ **algebraically**.

SCORE: ______ / 4 PTS

$$\underbrace{\lim_{x \to -\infty} \frac{e^{2x}-1}{e^{2x}+1}}_{1\frac{1}{2}} = \underbrace{\frac{0-1}{0+1}}_{1\frac{1}{2}} = \underbrace{-1}_{1}$$

For this question, you may use the formulae for $\frac{d}{dx}\sinh x$ and/or $\frac{d}{dx}\cosh x$ without proving them. **SCORE: ______ / 7 PTS**

If you need to use the formula for the derivative of any other hyperbolic function, you must prove it.

[a] **Without using the logarithmic formula for $\sinh^{-1} x$**, prove the formula for $\frac{d}{dx}\sinh^{-1} x$.

$y = \sinh^{-1} x$

$\sinh y = x$ (1)

$(\cosh y)\frac{dy}{dx} = 1$ (1)

$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+x^2}}$

(1) (1)

$\cosh^2 y - \sinh^2 y = 1$ (1)

$\cosh y = \sqrt{1+\sinh^2 y}$

SINCE $\cosh y > 0$ FOR ALL y

[b] **Without using the exponential formula for sech x**, prove the formula for $\frac{d}{dx}\operatorname{sech} x$.

$\frac{d}{dx}\frac{1}{\cosh x} = \frac{0 \cdot \cosh x - 1 \cdot \sinh x}{\cosh^2 x} = \frac{-\sinh x}{\cosh^2 x} = -\operatorname{sech} x \tanh x$

OR (1) (1)

OR

$\frac{d}{dx}(\cosh x)^{-1} = -(\cosh x)^{-2}\sinh x = -\operatorname{sech} x \tanh x$

Find $\dfrac{d}{dx}\dfrac{\tanh^{-1}(x^3)}{x^2}$. Simplify your final answer as a single fraction. **SCORE: ______ / 4 PTS**

<u>You may use the derivatives of any hyperbolic or inverse hyperbolic functions from your textbook without proving them.</u>

$$1 \quad \frac{\left(\frac{1}{1-(x^3)^2}\cdot 3x^2\right)x^2 - 2x\tanh^{-1}x^3}{(x^2)^2} \quad 1 \quad 1$$

$$= \frac{\frac{3x^{\cancel{4}\,3}}{1-x^6} - 2x\tanh^{-1}x^3}{x^{\cancel{4}\,3}}$$

$$= \boxed{\frac{3x^3 - 2(1-x^6)\tanh^{-1}x^3}{x^3(1-x^6)}} \quad 1$$

If $\tanh x = -\frac{2}{3}$, find $\sinh x$. **SCORE: ______ / 4 PTS**

$$1 - \tanh^2 x = \operatorname{sech}^2 x \quad (1)$$

$$1 - \frac{4}{9} = \operatorname{sech}^2 x$$

$$\frac{5}{9} = \operatorname{sech}^2 x$$

$$\operatorname{sech} x = \frac{\sqrt{5}}{3} \quad (1)$$

SINCE $\operatorname{sech} x > 0$ FOR ALL x

$$\cosh x = \frac{1}{\operatorname{sech} x} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5} \quad (\tfrac{1}{2})$$

EITHER IS OK

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$-\frac{2}{3} = \frac{\sinh x}{\frac{3\sqrt{5}}{5}} \quad (1)$$

$$\sinh x = -\frac{2}{3} \cdot \frac{3\sqrt{5}}{5} = -\frac{2\sqrt{5}}{5} \quad (\tfrac{1}{2})$$

Prove the logarithmic formula for $\tanh^{-1} x$ given in your textbook. **SCORE: ______ / 5 PTS**

NOTE: This is NOT a question about derivatives.

$$y = \tanh^{-1} x$$

$$\tanh y = x$$

$$\underbrace{\frac{e^{2y}-1}{e^{2y}+1} = x}_{} \quad 1$$

$$\underbrace{e^{2y} - 1 = xe^{2y} + x}_{} \quad 1$$

$$e^{2y} - xe^{2y} = 1 + x$$

$$\underbrace{(1-x)e^{2y} = 1 + x}_{} \quad 1\tfrac{1}{2}$$

$$e^{2y} = \frac{1+x}{1-x}$$

$$\underbrace{2y = \ln\frac{1+x}{1-x}}_{} \quad 1$$

$$y = \underbrace{\frac{1}{2}\ln\frac{1+x}{1-x}}_{\frac{1}{2}} = \tanh^{-1} x$$